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1991 J. Phys. A: Math. Gen. 24 L785

(http://iopscience.iop.org/0305-4470/24/14/006)

J. Phys. A: Math. Gen. 24 (1991) L785–L787. Printed in the UK

LETTER TO THE EDITOR

Symmetries of a nonlinear equation in plasma physics

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Received 18 April 1991

Abstract. The Lie symmetry vector fields are derived for a nonlinear equation in plasma physics.

For a collisionless plasma of cold ions and warm electrons, the basic system of partial differential equations may be given as follows [1]

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)=0 \qquad \text{(equation of continuity)} \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=E \qquad \text{(equation of motion)} \tag{2}$$

$$\frac{\partial n_e}{\partial x}=-n_e E \qquad \text{(balance of pressure and electric force)} \tag{3}$$

$$\frac{\partial E}{\partial x}=n-n_e \qquad \text{(the Poisson equation)} \tag{4}$$

where n and n_e denote the density of ions and electrons, respectively, u is the flow velocity of the ions, and E is the electric field. All these quantities are dimensionless. The inertia term is neglected because of the small mass of electrons. We can eliminate n and E from system (1)–(4). We obtain

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{n_e}\frac{\partial n_e}{\partial x}=0 \tag{5}$$

$$\frac{\partial n_e}{\partial t}+\frac{\partial}{\partial x}(n_e u)+\frac{\partial P}{\partial x}=0 \tag{6}$$

where

$$P=-\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left(\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right). \tag{7}$$

We now show that the Korteweg-de Vries equation is included in these equations under certain approximations. We introduce the transformation

$$\xi(x,t)=\varepsilon^{1/2}(x-t) \qquad \eta(x)=\varepsilon^{3/2}x \tag{8}$$

$$u(\xi(x,t),\eta(x))=u(x,t) \qquad n_e(\xi(x,t),\eta(x))=n_e(x,t) \tag{9}$$

0305-4470/91/140785+03\$03.50

and apply the formal expansion (reductive perturbation method)

$$u(\xi, \eta) = \varepsilon u^{(1)}(\xi, \eta) + \varepsilon^2 u^{(2)}(\xi, \eta) + \ldots \tag{10}$$

$$n_e(\xi, \eta) = 1 + \varepsilon n_e^{(1)}(\xi, \eta) + \varepsilon^2 n_e^{(2)}(\xi, \eta) + \ldots. \tag{11}$$

Then we find that

$$u^{(1)} = n_e^{(1)}$$

and

$$\frac{\partial u^{(1)}}{\partial \eta} + u^{(1)} \frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial^3 u^{(1)}}{\partial \xi^3} = 0 \tag{12}$$

$$\frac{\partial n_e^{(1)}}{\partial \eta} + n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{\partial^3 n_e^{(1)}}{\partial \xi^3} = 0. \tag{13}$$

Thus we see that u, the ion-fluid velocity, and n_e, the electron density, obey the same Korteweg-de Vries equation, and move with the same phase, $u^{(1)} = n_e^{(1)}$. The nonlinear term $u^{(1)}\partial u^{(1)}/\partial \xi$ comes from the interaction of ions with electrons affecting ions themselves, and $n_e^{(1)}\partial n_e^{(1)}/\partial \xi$ expresses a similar effect on electrons through the interaction with ions.

It is well known that the Korteweg-de Vries equation admits an infinite hierarchy of Lie-Bäcklund vector fields and an infinite hierarchy of conservation laws. Moreover, it admits a Lax representation, an auto-Bäcklund transformation and passes the Painlevé test [2, 3]. The Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{14}$$

has the following Lie symmetry vector fields

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t} \qquad t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \qquad x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

For our study of the symmetry vector fields of system (1)-(4) we adopt the jet bundle formalism. From system (1)-(4) we obtain the submanifolds [4]

$$F_1 \equiv n_t + n_x u + n u_x = 0 \tag{15}$$

$$F_2 \equiv u_t + u u_x - E = 0 \tag{16}$$

$$F_3 \equiv n_{ex} + n_e E = 0 \tag{17}$$

$$F_4 \equiv E_x - n + n_e = 0 \tag{18}$$

and their differential consequences. Let V be a Lie symmetry vector field. Let V_v be the corresponding vertical vector field. Then the invariance condition of system (1)-(4) is given by

$$L_{V_v} F_j \triangleq 0 \qquad j = 1, \ldots, 4$$

where $\triangleq$ stands for the restriction to solutions of system (1)-(4). Let us first study the scale invariance of system (1)-(4). The ansatz for the Lie symmetry vector field describing the scale invariance is given by

$$S = c_1 x \frac{\partial}{\partial x} + c_2 t \frac{\partial}{\partial t} + c_3 u \frac{\partial}{\partial u} + c_4 n \frac{\partial}{\partial n} + c_5 n_e \frac{\partial}{\partial n_e} + c_6 E \frac{\partial}{\partial E}$$

where $c_1, \ldots, c_6$ are real constants. From the invariance condition we find that $c_1 = \ldots = c_6 = 0$. This means that system (1)–(4) does not admit a scale symmetry.

The ansatz for the Lie symmetry vector fields

$$V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + c\frac{\partial}{\partial u} + d\frac{\partial}{\partial n} + e\frac{\partial}{\partial n_e} + f\frac{\partial}{\partial E} \tag{19}$$

where $a, \ldots, f$ are functions of x, t, u, n, n_e, E, gives the symmetry vector fields

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t} \qquad t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

The first two symmetry vector fields are obvious, since the system (1)–(4) does not depend explicitly on t and x. The transformation group associated with the third vector field is given by

$$t'(x, t, \varepsilon) = t \tag{20}$$

$$x'(x, t, \varepsilon) = \varepsilon t + x \tag{21}$$

$$u'(x'(x, t), t'(x, t), \varepsilon) = \varepsilon + u(x, t) \tag{22}$$

$$n'(x'(x, t), t'(x, t), \varepsilon) = n(x, t) \tag{23}$$

$$n_e'(x'(x, t), t'(x, t), \varepsilon) = n_e(x, t) \tag{24}$$

$$E'(x'(x, t), t'(x, t), \varepsilon) = E(x, t). \tag{25}$$

We also studied the existence of Lie Bäcklund symmetry vector fields. In this case the coefficients $a, \ldots, f$ of the vector field given by (19) depend on x, t, u, n, n_e, E, $u_x, \ldots, E_{xx}$. We find that system (1)–(4) does not admit Lie–Bäcklund vector fields of this form. Finally we mention that system (1)–(4) does not pass the Painlevé test [2, 3]. These results indicate that system (1)–(4) is not completely integrable, although an approximation leads to a completely integrable system, namely the Korteweg-de Vries equation.

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